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ALMOST SINUSOIDAL OSCILLATIONS IN NONLINEAR SYSTEMS

Part III: Transient Phenomena

J. S. Schaffner

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Part III: Transient Phenomena

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ABSTRACT

This bulletin is the third in a series of three bulletins dealing with a number of oscillatory problems. In the first two bulletins emphasis was placed on steady-state oscillations, while in this bulletin the discussion is concerned with nonsteady-state or transient oscillations. The oscillatory phenomena discussed here are parametric excitation, synchronization, simultaneous oscillations and amplitude limitation by means of lamps. The reader should understand, however, that the methods presented in this bulletin can also be applied to a number of other oscillatory problems.

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FIGURES

1. INTRODUCTION

One of the most significant trends of the twentieth century is the departmentalization of science. The broad field of human experience has been divided into innumerable sections and subsections, each mastered only by a small group of specialists. One of the consequences of this division of science is that structural similarities between different fields are often overlooked. The recognition of these similarities can frequently be a great source of inspiration; a source closed to the specialist who fails to develop any interest outside his own field.

A typical example of such a similarity is the occurence of oscillatory phenomena in such diversified fields as electronics, aerodynamics, economics, biology, etc. In recent years a uniform theory of oscillations applicable to all of these fields has been developed. However, the main applications of this theory at present are in electrical and aeronautical engineering. Mathematically, oscillatory phenomena lead to nonlinear differential equations. Because of this, the theory of oscillations is often called "nonlinear mechanics."

Most important for the study of an oscillatory system are the steady-state oscillations. They may be stable or unstable, depending on whether the oscillator will or will not return to its original state after being subjected to a small disturbance (noise, etc.). An oscillator may have several possible steady-state oscillations. In this bulletin particular emphasis is placed on the nonsteady-state or transient oscillations. These terms, nonsteady-state and transient, are used synonomously. The study of these transient oscillations is often important for the complete understanding of the circuit behavior. For example, in a system having several possible steady-state oscillations, the transient study predicts which one of these will be reached from a given set of initial conditions. It also determines the manner in which oscillations will build up and describes the behavior of the system if a circuit parameter is changed abruptly. Generally, a study of the transient oscillations results in a more complete understanding of the oscillatory system.

One important problem to be overcome by the investigator is that of representing the transient oscillations. Analytical representation is not practicable, since, even if an approximate formula could be found, it would be so complicated that it could not be interpreted easily. The most common graphical method of representation makes use of a plane with time as the abscissa and the dependent variable as the ordinate. This method is, however, limited to the cases in which the differential equation can be reduced to

$$\frac{dx}{dt} = f\,(x,t)$$

This reduction is possible for only a few oscillators.

Autonomous systems with one degree of freedom can often be described by the differential equation:

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + g\left(x, \frac{dx}{dt}\right)x = 0 \tag{1.1}$$

The corresponding oscillations can be represented in a plane with x as the abscissa and dx/dt as the ordinate, the "phase-plane." Such a representation is shown in Chapter II.

No general method for the graphical representation of oscillations in more complicated systems is available. If it can be assumed, however, that the oscillations are almost sinusoidal, then a large class of oscillators can be represented in some sort of a phase plane. In this new plane, an oscillation with a certain amplitude, phase angle, etc., will correspond to a single point.

II. AUTONOMOUS OSCILLATIONS WITH ONE DEGREE OF FREEDOM

The study of the autonomous oscillators with one degree of freedom takes a rather special place in nonlinear mechanics insofar as these oscillators can be treated conclusively and in all generality by topological methods. Equation 1.1, corresponding to autonomous oscillators with one degree of freedom, can be transformed into two simultaneous differential equations of the first order by the substitution $dx/dt = y$

$$\frac{dy}{dt} = -f(x,y)\,y - g(x,y)\,x$$

$$\frac{dx}{dt} = y \tag{2.1}$$

The time can be eliminated from these two equations by dividing one by the other:

$$\frac{dy}{dx} = -f(x,y) - g(x,y)\frac{x}{y} \tag{2.2}$$

The solutions of this equation can be represented by curves in the x-y plane, which can be found by graphical construction (method of isoclines). If the system is at rest, then x and y are constant in time; that is,

$$\frac{dx}{dt} = \frac{dy}{dt} = 0$$

and the differential, dy/dx, at such a point, is indeterminate (singular point).

The x-y plane may also contain a number of closed curves, or "limit cycles," corresponding to periodic solutions of Eq. 1.1. The singular points and the limit cycles determine the structure of the x-y plane. The state of the oscillator is completely described at any one time by a point in the x-y plane. If the system is not disturbed, then this "representative" point will move along a curve defined by Eq. 2.2.

Typical oscillatory systems that can be treated by this method are, for example, the ordinary triode oscillator, the Prony-brake and the multivibrator. It is possible but impractical to treat more complicated systems by this method because spaces with three or more dimensions are required to describe them.

III. ALMOST SINUSOIDAL OSCILLATIONS

A large group of oscillatory phenomena can be represented in a plane entirely different from that discussed in Chapter II provided that it can be assumed that the oscillations are almost sinusoidal. In electrical circuits, for example, oscillations are nearly sinusoidal if the quality factors (Q's) of the resonant circuits are high. Then if it is assumed that the oscillations are purely sinusoidal, the circuits can be described completely by a few parameters; for example, by the amplitude of oscillation, the phase angle relative to an external sinusoidal voltage, etc. For steady-state oscillations, these parameters remain constant; for transient oscillations, they will change. A representation in a plane is possible if the system can be described by two parameters only, say X and Y. When one of these parameters is a phase angle, the plane lies on the surface of a cylinder so that the lines, $\phi = 0$ and $\phi = 2\pi$, are identical. Many oscillatory phenomena can be described by two parameters and can, hence, be represented in this plane.

The rate of change of X and Y is determined by the state of the system and, hence, depends on X and Y only.

$$\frac{dX}{dt} = f_1(X,Y)$$
$$\frac{dY}{dt} = f_2(X,Y) \tag{3.1}$$

Equations 3.1 can be obtained by various analytical and experimental methods. The method used in this bulletin is that of equivalent linearization. The mathematical formulation of this method has been described extensively in Buls. 395 and 400, and hence is not presented here.

The time can be eliminated from Eqs. 3.1 by dividing one by the other.

$$\frac{dX}{dY} = \frac{f_1(X,Y)}{f_2(X,Y)} \tag{3.2}$$

The solutions of this differential equation can be represented as curves in the X-Y plane. If the system is not disturbed, then the "representative point," corresponding to an oscillation with parameters X and Y, will move along one of these curves. For steady-state oscillations, X and Y remain constant in time.

$$\frac{dX}{dt} = \frac{dY}{dt} = 0 \tag{3.3}$$

This corresponds to a singular point in the X-Y plane. The symbols, X_0 and Y_0, are used for the parameters of the steady-state oscillation.

A singular point and the corresponding steady-state oscillation are stable if the oscillation and the corresponding point will return to their original position after a small disturbance. In order to discuss the behavior of the system in the neighborhood of the steady-state oscillations, it is necessary to expand Eq. 3.1 around X_0 and Y_0. Letting

$$\delta X = X - X_0$$
$$\delta Y = Y - Y_0$$

then the first terms of the expansion are

$$\frac{d\,(\delta X)}{dt} = \frac{\partial f_1}{\partial X}\,\delta X + \frac{\partial f_1}{\partial Y}\,\delta Y + \ldots$$
$$\frac{d\,(\delta Y)}{dt} = \frac{\partial f_2}{\partial X}\,\delta X + \frac{\partial f_2}{\partial Y}\,\delta Y + \ldots \tag{3.4}$$

It is assumed that δX and δY are small and therefore that the higher terms of this expansion can be neglected. Equation 3.4 is a set of two simultaneous linear differential equations which can be solved by the usual methods. The variables δX and δY will approach zero from any initial conditions in the neighborhood of the steady-state oscillations provided that both the roots κ_1 and κ_2 of Eq. 3.5 have negative real parts.

$$\begin{vmatrix} \frac{\partial f_1}{\partial X} - \kappa & \frac{\partial f_1}{\partial Y} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} - \kappa \end{vmatrix} = 0 \tag{3.5}$$

Necessary and sufficient conditions for this to exist are that:

$$\frac{\partial f_1}{\partial X} + \frac{\partial f_2}{\partial Y} < 0$$
$$\frac{\partial f_1}{\partial X}\frac{\partial f_2}{\partial Y} - \frac{\partial f_1}{\partial Y}\frac{\partial f_2}{\partial X} > 0 \tag{3.6}$$

Like the limit cycles of the x-y plane described in Chapter II, the X-Y plane may contain closed curves which are called "drift curves." Physically, these drift curves correspond to oscillations with periodically varying amplitude.

IV. PARAMETRIC EXCITATION OF NONLINEAR SYSTEMS

Oscillations in resonant circuits can be generated in several different ways; for example, by an external sinusoidal force or by a negative resistance. Another possibility is the excitation of the circuit by means of the periodic variation of a circuit parameter. This excitation is particularly strong if the capacitance or inductance of the circuit is varied with a frequency approximately twice that of the resonant frequency of the circuit. The amplitude of the oscillations so generated will build up to a value determined by the nonlinearities of the system.

Assume for example that in the circuit of Fig. 4.1, the distance d between the plates of the capacitance C is varied according to

$$d = d_0 + d_1 \cos 2\omega t \tag{4.1}$$

where $d_1 \ll d_0$ and ω is close to the resonant frequency of the circuit. Then if the resistance R is sufficiently small, oscillations with a frequency ω will build up. As is shown later, they may reach a stationary value if either R or L is a function of the current i (nonlinear resistance or inductance).

In the first part of this chapter, it is assumed that R and L are constant; $R = R_0$, $L = L_0$. The capacitance C corresponding to Eq. 4.1 is:

$$C = \frac{C_0}{1 + \gamma \cos 2\omega t}$$

where $\gamma = d_1/d_0 \ll 1$.

The circuit of Fig. 4.1 can be transformed to an "equivalent linear circuit" by the method of equivalent linearization described in Buls. 395 and 400. The first step is to split the capacitance C into two parts, C_1 and C_{10}, as shown in Fig. 4.2.

$$\begin{aligned} C_1 &= \frac{\omega_0^2}{\omega^2} C_0 \\ C_{10} &= \frac{C_0}{1 - \dfrac{\omega^2}{\omega_0^2} + \gamma \cos 2\omega t} \end{aligned} \tag{4.2}$$

where $\omega_0^2 = 1/L_0C_0$ is the resonant frequency of the system. It was assumed that $\omega \cong \omega_0$; therefore C_{10} is very large.

For a large C_{10} and a small R_0, the current i is approximately

$$i \cong I \cos(\omega t + \phi)$$

The amplitude I and the phase ϕ will of course change, but over a few cycles they will remain practically constant. The voltage v across R_0 and C_{10} is (Fig. 4.2):

$$\begin{aligned} v &= \frac{1}{C_{10}} \int i dt + iR_0 \\ &= I \frac{1 - \omega^2/\omega_0^2 - \gamma/2 \cos 2\phi}{\omega C_0} \sin(\omega t + \phi) \\ &+ I\left(R_0 + \frac{\gamma}{2\omega C_0} \sin 2\phi\right) \cos(\omega t + \phi) \\ &+ I \frac{\gamma}{2\omega C_0} \sin(3\omega t + \phi) \end{aligned}$$

Due to the component of v at frequency 3ω a small current at this frequency will flow through L_0 and C_1. Since this component is very small, however, it can be neglected. As is shown, the components at frequency ω will cause a change of I and ϕ.

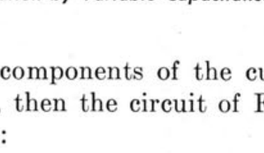

Fig. 4.1. General Oscillatory Circuit with Excitation by Variable Capacitance

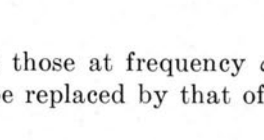

Fig. 4.2. Oscillatory Circuit with Variable Capacitance Excitation. R and L Constant

If all components of the current except those at frequency ω are neglected, then the circuit of Fig. 4.2 can be replaced by that of Fig. 4.3 where:

$$\begin{aligned} R_e &= R_0 + \frac{\gamma}{2\omega C_0} \sin 2\phi \\ C_e &= \frac{C_0}{1 - \omega^2/\omega_0^2 - \gamma/2 \cos 2\phi} \end{aligned} \tag{4.3}$$

The circuit of Fig. 4.3 is linear and can be discussed by the method of linear analysis.

Fig. 4.3. Equivalent Linearized Circuit of Fig. 4.2

The energy stored in the oscillator, $L_0I^2/2$, will remain approximately constant over one cycle. Over a longer period of time, it will change slowly since part of it is dissipated in the resistance R_e.

$$\frac{d\,(L_0I^2/2)}{dt} = \frac{I^2}{2}R_e$$

or

$$\frac{dI}{dt} = -\frac{I}{2L_0}\left(R_0 + \frac{\gamma}{2\omega C_0}\sin 2\phi\right) \tag{4.4}$$

Similarly, the phase angle ϕ will change slowly.

$$\omega + \frac{d\phi}{dt} = \frac{1}{\sqrt{L_0\dfrac{C_1C_e}{C_1 + C_e}}}$$

or

$$\frac{d\phi}{dt} = \frac{1}{2}\,\omega\,\frac{C_0}{C_e}$$

$$\cong \Delta\omega - \frac{\gamma}{4}\,\omega\cos 2\phi \tag{4.5}$$

where

$$\Delta\omega = \omega_0 - \omega$$

The time can be eliminated from these two equations by dividing one by the other:

$$\frac{d\phi}{dI} = \frac{\Delta\omega - (\gamma/4)\omega\cos 2\phi}{-(I/2L_0)\left[R_0 + \dfrac{\gamma}{2\omega C_0}\sin 2\phi\right]}$$

$$= \frac{z - \cos 2\phi}{-I\,(\rho + \sin 2\phi)} \tag{4.6}$$

where

$$z = 4\,\frac{\Delta\omega}{\omega\gamma}$$

$$\rho = \frac{2R_0\omega C_0}{\gamma}$$

z is proportional to the "detuning" of the resonant circuit.

The trajectories in the I-ϕ plane corresponding to Eq. 4.6 permit graphical representation of the transient solutions. They can be obtained either by the method of isoclines or by direct integration. Typical groups of trajectories are shown in Fig. 4.4.

Steady-state oscillations exist if $dI/dt = d\phi/dt = 0$. This corresponds to singular points in the I-ϕ plane. The stability of these singular points and of the corresponding steady-state oscillations can be determined from the behavior of the trajectories in their neighborhood. It can be seen that they are unstable, because after a small disturbance of ϕ the amplitude I will increase or decrease indefinitely.

In order to obtain stable steady-state oscillations in the circuit of Fig. 4.1, it is necessary that the circuit contain a nonlinear inductance or a nonlinear resistance.

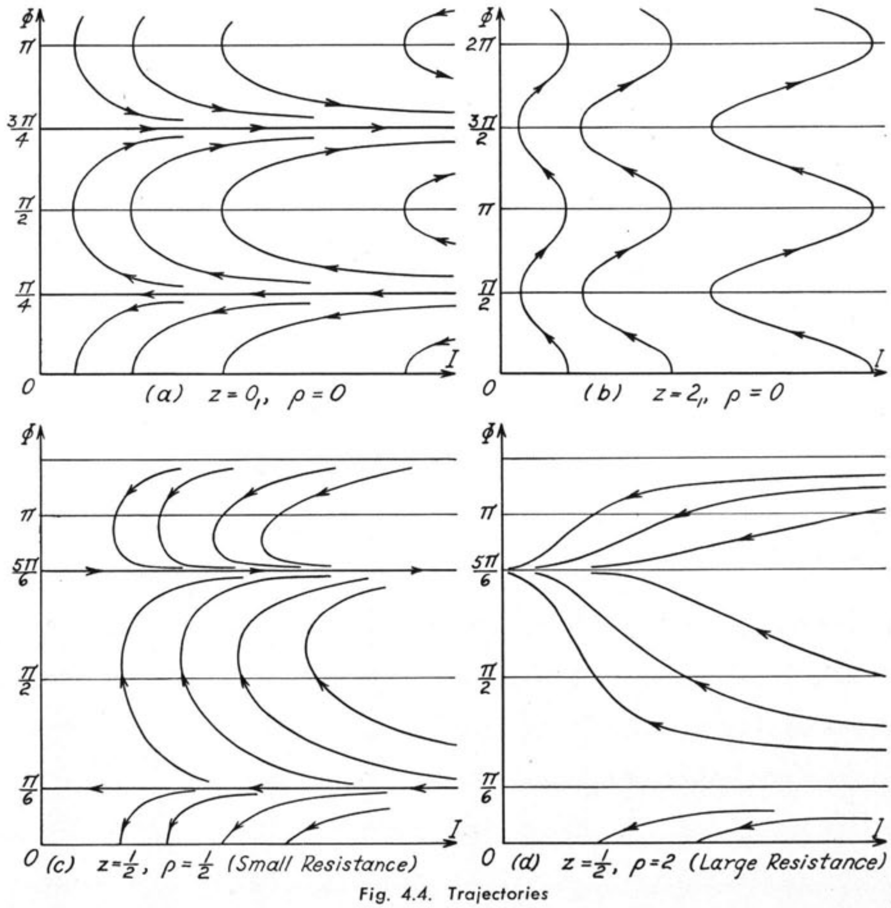

Fig. 4.4. Trajectories

Consider first the case where the inductance L of Fig. 4.1 is a function of the current i passing through it:

$$L = L_0 (1 + \epsilon i^2) \qquad \epsilon i^2 \ll 1$$

The oscillatory curcuit corresponding to Fig. 4.1 with a nonlinear inductance is shown in Fig. 4.5. It is again assumed that

$$i = I \cos (\omega t + \phi)$$

The voltage v is then:

$$\begin{aligned} v &= \frac{d}{dt}(L_0 \epsilon i^3) + R_0 i + \frac{1}{C_{10}} \int i dt \\ &= I \left\{ -\frac{3}{4}\epsilon L_0 I^2 \omega + \frac{1 - \omega^2/\omega_0^2 - \gamma/2 \cos 2\phi}{\omega C_0} \right\} \sin (\omega t + \phi) \\ &+ I \left(R_0 + \frac{\gamma}{2\omega C_0} \cos 2\phi \right) \cos (\omega t + \phi) \\ &+ \text{terms at frequency } 3\omega. \end{aligned}$$

The terms at the frequency 3ω are again neglected. The equivalent conductance and capacitance are:

$$\begin{aligned} R_e &= R_0 + \frac{\gamma}{2\omega C_0} \sin 2\phi \\ C_e &= \frac{C_0}{1 - \omega^2/\omega_0^2 \cos 2\phi - \frac{3}{4}\epsilon I^2} \end{aligned} \tag{4.7}$$

The differential equation corresponding to Eq. 4.6 is:

$$\begin{aligned} \frac{d\phi}{dI} &= \frac{\Delta\omega - (\gamma/4)\omega \cos 2\phi - \frac{3}{8}\epsilon I^2 \omega}{-(I/2L_0)[R_0 + (\gamma/2\omega C_0)\sin 2\phi]} \\ &= -\frac{z - \cos 2\phi - \alpha I^2}{I(\rho + \sin 2\phi)} \end{aligned}$$

where

$$\begin{aligned} z &= 4 \frac{\Delta\omega}{\omega_0 \gamma} \\ \rho &= \frac{2R_0 \omega C_0}{\gamma} \\ \alpha &= \frac{3}{2} \frac{\epsilon}{\gamma} \end{aligned}$$

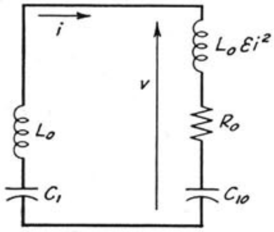

Fig. 4.5. Oscillatory Circuit with Variable Capacitance Excitation. Inductance Nonlinear

For a steady-state oscillation it is necessary that

$$\frac{dI}{dt} = \frac{d\phi}{dt} = 0$$

or, for $I \neq 0$,

$$\begin{aligned} z - \alpha I^2 - \cos 2\phi &= 0 \\ \rho + \sin 2\phi &= 0 \end{aligned} \tag{4.8}$$

The symbols I_0 and ϕ_0 are used for the values of I and ϕ that satisfy Eqs. 4.8. For each set of z, α, ρ there are at most two independent solutions:

$$I_0^2 = \frac{z - \sqrt{1 - \rho^2}}{\alpha} \qquad (\cos 2\phi > 0)$$

and

$$I_0^2 = \frac{z + \sqrt{1 - \rho^2}}{\alpha} \qquad (\cos 2\phi < 0)$$

Only one of these solutions is stable, however.

The characteristic equation corresponding to Eqs. 4.4 and 4.5 is

$$\begin{vmatrix} -\dfrac{I_0}{2L_0}\dfrac{\partial R_e}{\partial I} - \kappa & -\dfrac{I_0}{2L_0}\dfrac{\partial R_e}{\partial \phi} \\ +\dfrac{\omega C_0}{2}\dfrac{\partial}{\partial I}\left(\dfrac{1}{C_e}\right) & \dfrac{\omega C_0}{2}\dfrac{\partial}{\partial \phi}\left(\dfrac{1}{C_e}\right) - \kappa \end{vmatrix} = 0$$

Necessary and sufficient conditions for stability are that

$$-\frac{I_0}{2L_0}\frac{\partial R_e}{\partial I} + \frac{\omega C_0}{2}\frac{\partial}{\partial \phi}\left(\frac{1}{C_e}\right) < 0$$

$$-\frac{\partial R_e}{\partial I}\frac{\partial}{\partial \phi}\left(\frac{1}{C}\right) + \frac{\partial R_e}{\partial \phi}\frac{\partial}{\partial I}\left(\frac{1}{C_e}\right) > 0$$

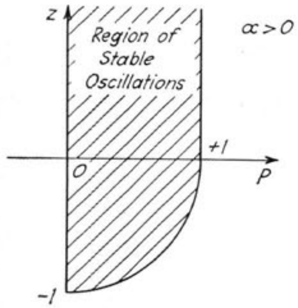

Fig. 4.6. Region of Stable Oscillations for $a > 0$

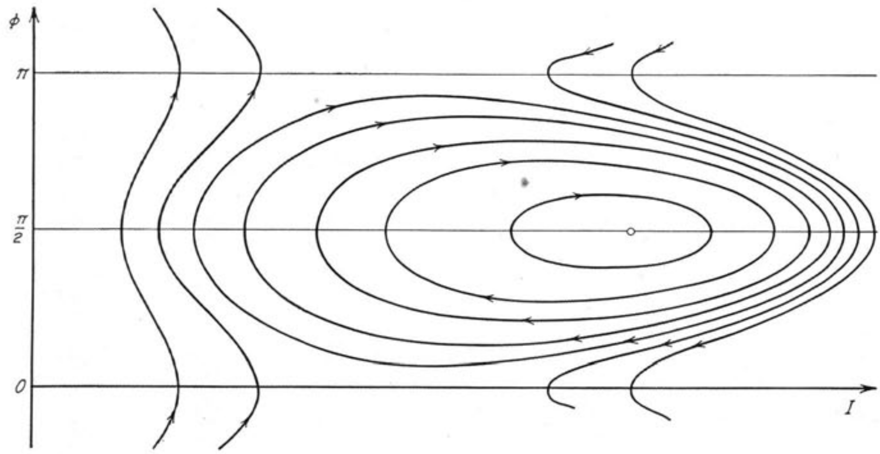

Fig. 4.7. Trajectories for $\rho = 0$, $z = 2$ (Large Detuning)

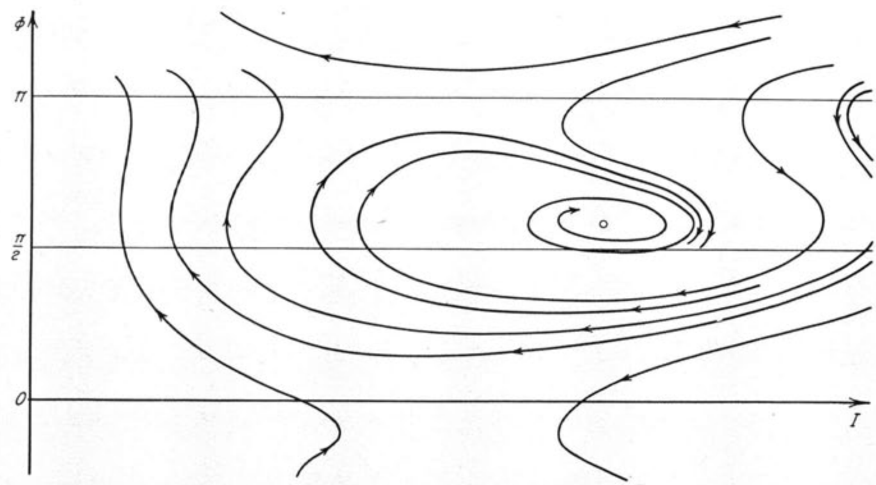

Fig. 4.8. Trajectories for $\rho = 1/2$, $z = 2$ (Large Detuning)

or, in terms of ϕ that

$$\sin 2\phi < 0$$
$$\alpha \cos 2\alpha < 0$$

The first of these inequalities is satisfied if $\rho > 0$ (Eq. 4.8). The second inequality shows that, of the two solutions for I_0, only one is stable. For $\alpha > 0$ the solution with $\cos 2\phi < 0$ is stable; for $\alpha < 0$ the solution with $\cos 2\phi > 0$ is stable. The regions of stable oscillations can be represented in the ρ-z plane (Fig. 4.6). Additional boundaries of the region correspond to: $I^2 > 0$ and $\rho = -\sin 2\phi < 1$. The region does not depend on the magnitude of α as long as α remains finite. If ρ is small, then the inductance may arbitrarily be increased or decreased depending on the sign of α. This result, however, is due to the many simplifications made for this calculation; for example, with regard to the dependence of the inductance, L, on the current, i. In any physical system, stable oscillations will occur only within a finite range of variation of the inductance. Figures 4.7 and 4.8 show two sets of trajectories corresponding to systems with nonlinear inductance. For Fig. 4.7 the system is assumed to contain no damping ($\rho = 0$) and large detuning ($z = 2$). The singularity is of the center-type and all trajectories are also drift curves. The case $\rho = 0$ cannot occur in a physical system, however, since a small amount of damping is necessarily contained in any system. Figure 4.8 corresponds to small damping ($\rho = \frac{1}{2}$) and large detuning ($z = 2$). The stable singularity is a focal point which, however, cannot be reached by the system from rest ($I = 0$) unless the detuning is considerably smaller. This corresponds to hard self-excitation of the ordinary oscillator.

Stable steady-state oscillations may also exist if the circuit contains a resistance that is a function of the current i passing through it. (See Fig. 4.9.)

$$R = R_0 (1 + \mu i^2)$$

R_0 and μ may be positive or negative. If $R_0 < 0$, $\mu > 0$, and $\gamma = 0$ (C is constant in time), then the circuit of Fig. 4.1 will oscillate with an amplitude

$$I^2 = -\frac{4}{3\mu} \tag{4.10}$$

which result is used later.

It can be shown (Fig. 4.10) that the equivalent capacitance and resistance are

$$C_e = \frac{C_0}{1 - \omega^2/\omega_0^2 - (\gamma/2 \cos)2\phi}$$

$$R_e = R_0\left(1 + \frac{3}{4}\mu I^2\right) + \frac{\gamma}{2} \sin 2\phi$$

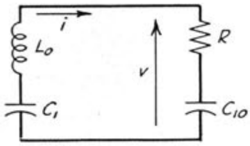

Fig. 4.9. Oscillatory Circuit with Variable Capacitance Excitation. Resistance Nonlinear

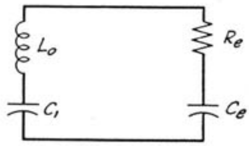

Fig. 4.10. Equivalent Linearized Circuit of Fig. 4.9

and that the differential equation corresponding to Eq. 4.6 is

$$\frac{d\phi}{dI} = \frac{z - \cos 2\phi}{I\,(\rho\,(1 + 3/4\,\mu I^2) + \sin 2\phi)} \tag{4.11}$$

where

$$z = 4\,\frac{\Delta\omega}{\omega\gamma}$$

$$\rho = \frac{2R_0\omega C_0}{\gamma}$$

The conditions for steady state oscillations are

$$z - \cos 2\phi = 0$$

$$\rho\,(1 + 3/4\mu I^2) + \sin 2\phi = 0 \tag{4.12}$$

and the conditions for stability

$$\sin 2\phi < 0$$

$$\rho\mu > 0$$

Figure 4.11 shows the region in the ρ-z plane where stable oscillations are possible. Of these stable oscillations two different cases must be distinguished — $\rho < 0$ and $\rho > 0$.

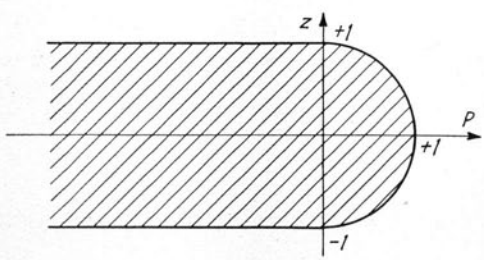

Fig. 4.11. Region of Stable Oscillations

(a) $0 < \rho < 1 \qquad \mu > 0$

The amplitude of oscillation I can be found from Eqs. 4.12. The dependence of the amplitude on detuning is shown in Fig. 4.12. Outside the "region of synchronization" the amplitude of oscillation is zero. A group of trajectories for $z = 1/2$, $\rho = 1/3$, $\mu = 1$ is shown in Fig. 4.13.

(b) $\rho < 0 \qquad \mu < 0$

The dependence of the amplitude on the detuning for this case is shown in Fig. 4.12*b*. Inside the region of synchronization there is little difference between this and case *a*, but outside the region the amplitude does not drop to zero. This is due to the fact that even without parametric excitation the circuit will oscillate.

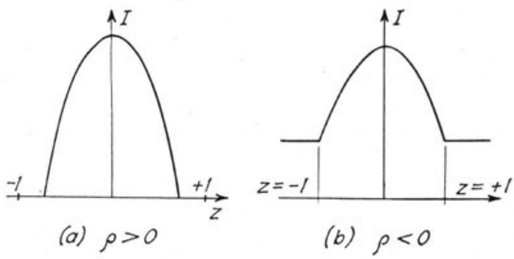

Fig. 4.12. Dependence of the Amplitude I on Detuning

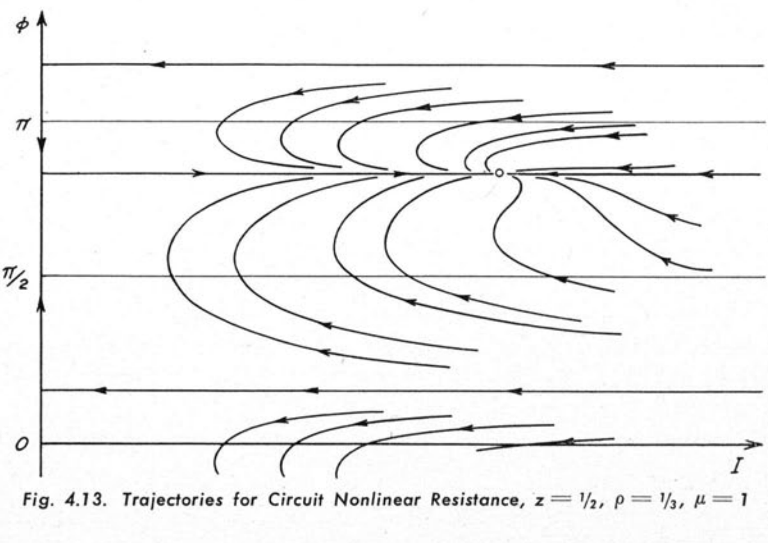

Fig. 4.13. Trajectories for Circuit Nonlinear Resistance, $z = 1/2$, $\rho = 1/3$, $\mu = 1$

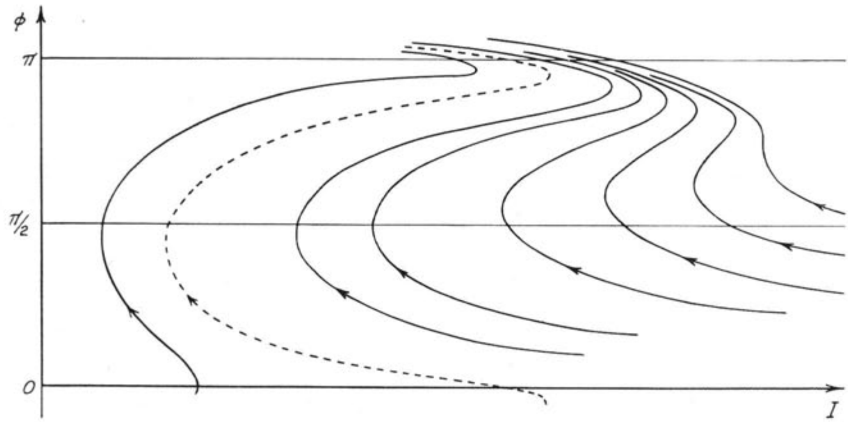

Fig. 4.14. Trajectories for $z = 1.05$, $\rho = -1/2$, $\mu = -1$

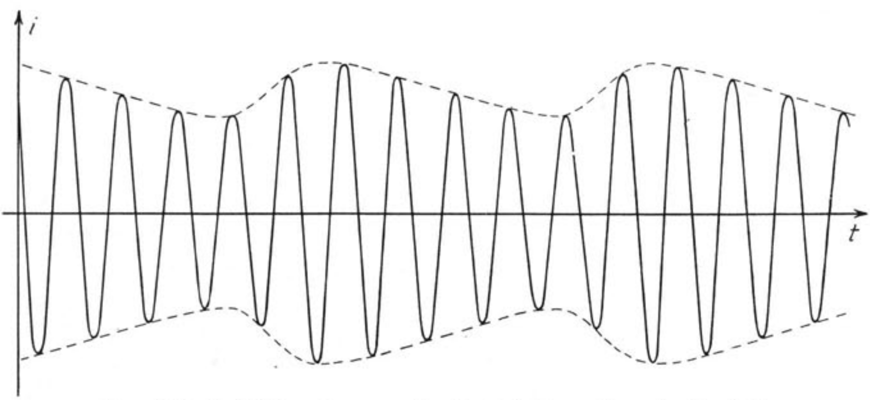

Fig. 4.15. Oscillations Corresponding to Drift Curve Shown in Fig. 4.14

Figure 4.14 shows the trajectories for $z = 1.05$. Because this is outside the "region of synchronization," no singular points exist. There exists, however, one (and only one) drift curve shown dotted in Fig. 4.13. Figure 4.15 shows the oscillation corresponding to that curve. This figure shows a good agreement with some experimental results published by W. L. Barrow.[(6)*]

*Parenthesized superscripts refer to correspondingly numbered entries in the Bibliography.

V. SYNCHRONIZATION

In this chapter the oscillatory circuit of Fig. 5.1 is discussed. This circuit is the equivalent of an ordinary plate-tuned oscillator with an external sinusoidal synchronizing voltage v_s. When v_s is zero, the circuit will oscillate with a frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The frequency of oscillation ω for $v_s \neq 0$ is not necessarily ω_0. As the frequency of the external voltage ω_s is varied and becomes approximately equal to ω_0, ω will suddenly change from ω_0 to ω_s. This phenomena is called locking, synchronization, or entrainment of frequency. It occurs not only if $\omega_s \cong \omega_0$ but also if $\omega_s/\omega_0 = p/q$, where p and q are "small integers." This phenomenon is called subharmonic synchronization and has been discussed in detail in Bul. 400.

If it is assumed that $v_s = V_s \cos \omega_s t$, then the voltage v_1 between points A and B is approximately sinusoidal, particularly if the losses in the resonant circuits are small

$$v_1 \cong V \cos (\omega t + \phi)$$

It was shown in Bul. 400 that the circuit of Fig. 5.1 can then be reduced to that of Fig. 5.2 in which the non-linear element is replaced by an "equivalent network." The differential equations for V and ϕ for this circuit are

$$\frac{dV}{dt} = -\frac{V}{2C}[G + G_e] \tag{5.1}$$

$$\frac{d\phi}{dt} = \Delta\omega - \frac{1}{2}\omega_0 \frac{C_e}{\phi} \tag{5.2}$$

where

$$\Delta\omega = \omega_0 - \frac{g}{p}\omega_s$$

These equations determine the transient behavior of the oscillator. Dividing Eq. 5.2 by Eq. 5.1 eliminates the time

$$\frac{d\phi}{dV} = -\frac{2\Delta\omega C - \omega_0 C_e}{v\,(G + G_e)} \tag{5.3}$$

This equation permits representation in a plane with V and ϕ as coordinates, the V-ϕ plane. Steady-state oscillations ($dV/dt = d\phi/dt = 0$) correspond to singular points. Besides singular points, the V-ϕ plane

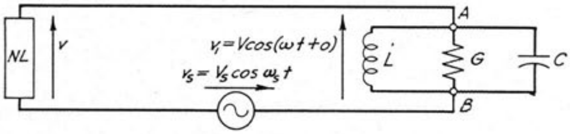

Fig. 5.1. Equivalent Circuit of a Tuned-Plate Oscillator with External Synchronizing Voltage

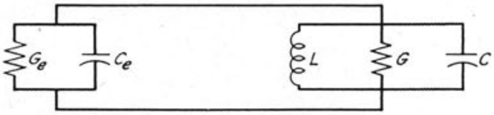

Fig. 5.2. Equivalent Linearized Circuit of Fig. 5.1

may also contain drift curves. They occur particularly if an oscillator is close to synchronization.[12]

As an example, the special case $\omega = 3\omega_s$ is presented here. The frequency of the oscillator is three times that of the external voltage (frequency-multiplication). It is assumed that the current i of Fig. 5.1 can be expressed by a third degree polynomial of the voltage v

$$i = \alpha v + \beta v^2 + \gamma v^3$$

The equivalent conductance and capacitance for $\omega = 3\omega_s$ and a third degree polynomial were calculated in Bul. 400. They are:

$$G_e = \alpha + \frac{3}{4}\gamma\,(V^2 + 2V_s^2) + \frac{\gamma}{4}\frac{V_s^3}{V}\cos\phi$$

$$C_e = \frac{\gamma}{4\omega_0}\frac{V_s^3}{V}\sin\phi \tag{5.4}$$

Using these two expressions, Eq. 5.3 may be written as

$$\frac{d\phi}{dV} - \frac{zV_0^2 + (V_s^3/V)\sin\phi}{3V\,(V - V_0^2) + V_s^3\cos\phi} \tag{5.5}$$

where

$$z = \frac{8\omega C}{\gamma V_0^2}\frac{\Delta\omega}{\omega}$$

$$V_0^2 = -\frac{4\,(G + \alpha)}{3\gamma} - 2V_s^2$$

Introducing the dimensionless parameters $X = V/V_0$ and X_s/V_0, Eq. 5.5 can be simplified to

$$\frac{d\phi}{dX} = -\frac{Xz + X_s^3\sin\phi}{X\,[3X\,(X^2 - 1) + X_s^3\cos\phi]} \tag{5.6}$$

The solutions of this differential equation can be represented as trajectories in the X-ϕ plane. The singular points in this plane are stable if they fall inside the region of stability in Fig. 5.3. Figure 5.4

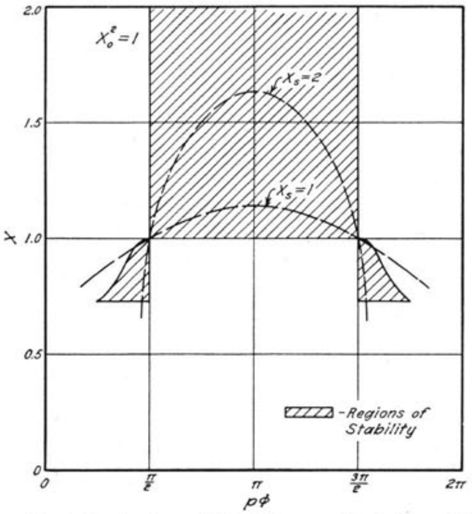

Fig. 5.3. Regions of Stability and Variation of X as a Function of $p\phi$ for $p = 3$, $q = 1$

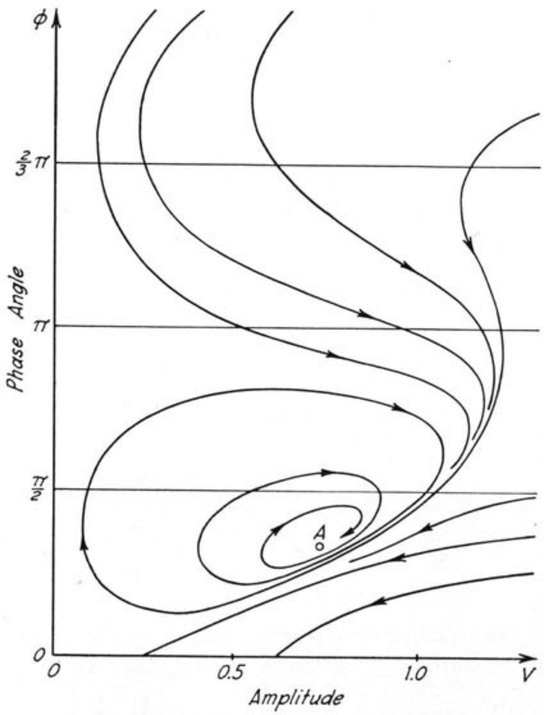

Fig. 5.4. Trajectories with Stable Singular Point

shows the family of trajectories in the X-ϕ plane corresponding to a stable singular point (point A). This point is a focus. Figure 5.5 shows the trajectories in the X-ϕ plane for an unstable singular point (point B). This figure also shows a drift curve corresponding to "near synchronization."

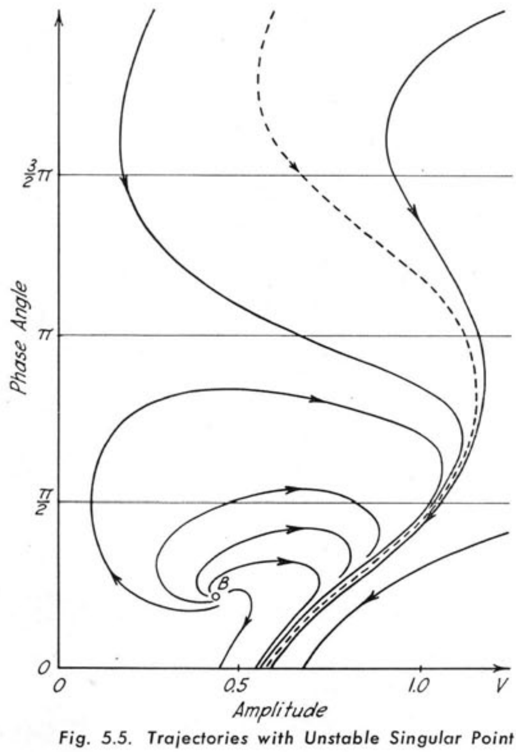

Fig. 5.5. Trajectories with Unstable Singular Point

VI. SIMULTANEOUS OSCILLATIONS

One of the most controversial and also one of the more interesting phases of nonlinear mechanics is that of simultaneous oscillations in circuits with two degrees of freedom. In recent years a number of papers dealing with this problem have been written and it has been proven both theoretically and experimentally that simultaneous oscillations are possible.[17, 18, 19]

For simultaneous oscillations, the voltage between points A and B of Fig. 6.1 is, approximately,

$$v_1 = V_1 \cos (\omega_1 t + \phi_1)$$

and the voltage between B and C is

$$v_2 = V_2 \cos (\omega_2 t + \phi_2) \tag{6.1}$$

where $\omega_1 \neq \omega_2$. A further distinction between ω_1 and ω_2 gives rise to the necessity for distinguishing between synchronous and asynchronous simultaneous oscillations. For synchronous oscillations the frequencies ω_1 and ω_2 are related by $p\omega_1 = q\omega_2$, where p and q are small integers, and comes about as a result of an effect of mutual synchronization. When the sum $(p+q)$ is large, the synchronization becomes so weak that it can no longer counteract the continuous disturbances caused by line fluctuations, noise, etc. For asynchronous oscillations no relation $p\omega_1 = q\omega_2$ is preserved over an appreciable length of time. Mathematically it is assumed in this case that the ratio ω_1/ω_2 is constant but irrational. Asynchronous simultaneous oscillations were discussed briefly in Bul. 395.

The oscillations in a circuit with two degrees of freedom can be represented in a four-dimensional phase space with $(v_1, dv_1/dt, v_2, dv_2/dt)$ as coordinates. For a successful treatment, however, it is imperative that the oscillations of any system be represented in a plane or at most in a three-dimensional space. The four dimensions of the present case can be reduced to three if it can be assumed that v_1 and v_2 are sinusoidal. The oscillations can then be represented in a space having as coordinates V_1, V_2 and, for synchronous oscillations, the angle $\phi = p\phi_1 - q\phi_2$ which is defined as the phase angle between v_1 and v_2. An oscillation which has amplitudes V_1, V_2 and phase angle ϕ corresponds then to a point in this space.

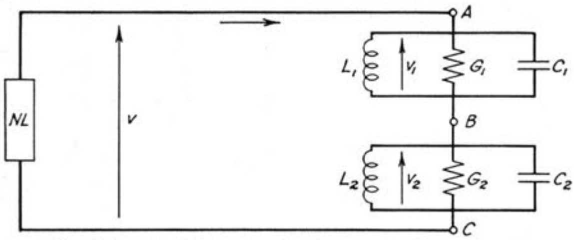

Fig. 6.1. Oscillator with Two Degrees of Freedom

The voltage v of Fig. 6.1 is

$$v = v_1 + v_2 = V_1 \cos (\omega_1 t + \phi_1) + V_2 \cos (\omega_2 t + \phi_2)$$

This voltage applied to the nonlinear element will produce a current

$$\begin{aligned} i &= I_{11} \cos (\omega_1 t + \phi_1) + I_{12} \sin (\omega_1 t + \phi_1) \\ &\quad + I_{21} \cos (\omega_2 t + \phi_2) + I_{22} \sin (\omega_2 t + \phi_2) \\ &\quad + \text{terms at frequencies other than } \omega_1 \text{ or } \omega_2. \end{aligned} \tag{6.2}$$

As shown in Bul. 395, Chapter I, the circuit of Fig. 6.1 can then be replaced by the two circuits of Fig. 6.2 where

$$\begin{aligned} G_{1e} &= \frac{I_{11}}{V_1}; & C_{1e} &= -\frac{I_{12}}{\omega_1 V_1} \\ G_{2e} &= \frac{I_{21}}{V_2}; & C_{2e} &= -\frac{I_{22}}{\omega_2 V_2} \end{aligned} \tag{6.3}$$

where the equivalent conductances and capacitances are functions of V_1, V_2, and ϕ.

If p and q are small integers, then the expression $\phi = p\phi_1 - q\phi_2$ is determined independent of a shift of the time axis. However, if either p or q is irrational, ϕ can still be adjusted arbitrarily close to any value

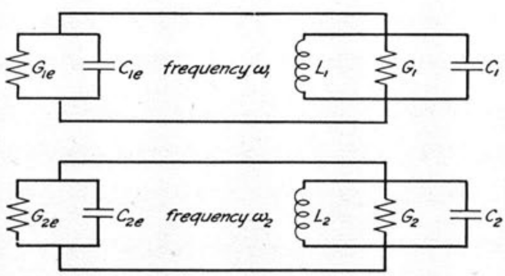

Fig. 6.2. Equivalent Circuit of Fig. 6.1

by a suitable shift of the time axis. Since this shift is a purely mathematical operation and cannot have any influence on the oscillator, it is clear that the equivalent impedances for asynchronous oscillations cannot be functions of ϕ.

The transient and steady-state behavior of the system can be calculated from the circuits of Fig. 6.2. The conductances of Fig. 6.2 will damp the oscillations and change the amplitudes. The capacitances C_1 and C_2 will change the frequencies, ω_1 and ω_2, and cause a shift of the phase angle, ϕ.

The corresponding differential equations are for asynchronous oscillations

$$\frac{dV_1}{dt} = -\frac{V_1}{2C_1}(G_1 + G_{1e})$$
$$\frac{dV_2}{dt} = -\frac{V_2}{2C_2}(G_2 + G_{2e}) \tag{6.5}$$

and for synchronous oscillations

$$\frac{dV_1}{dt} = -\frac{V_1}{2C_1}(G_1 + G_{1e}) \tag{6.6}$$

$$\frac{dV_2}{dt} = -\frac{V_2}{2C_2}(G_2 + G_{2e}) \tag{6.7}$$

$$\frac{d\phi}{dt} = \Delta\omega - \frac{p\omega_1}{2}\frac{C_e}{C_1} \tag{6.8}$$

where

$$\Delta\omega = \frac{p}{\sqrt{L_1C_1}} - \frac{q}{\sqrt{L_2C_2}}$$
$$C_e = C_{1e} - \frac{C_1}{C_2}C_{2e}$$

By dividing Eqs. 6.5 it is possible to eliminate the time between them, giving

$$\frac{dV_1}{dV_2} = \frac{V_1C_2}{V_2C_1}\frac{(G_1 + G_{1e})}{(G_2 + G_{2e})} \tag{6.9}$$

This differential equation can be solved by the methods of isoclines in a plane with V_1 and V_2 as coordinates. A point in this plane corresponds to an oscillation with the amplitudes, V_1 and V_2. If the system is not disturbed, then the representative point will move along a trajectory. Figures 6.3 and 6.4 show typical groups of trajectories. As shown in Bul. 395, stable simultaneous steady-state oscillations are not possible if the current and voltage across the nonlinear element are related by

$$i = \alpha v + \beta v^2 + \gamma v^3 \tag{6.10}$$

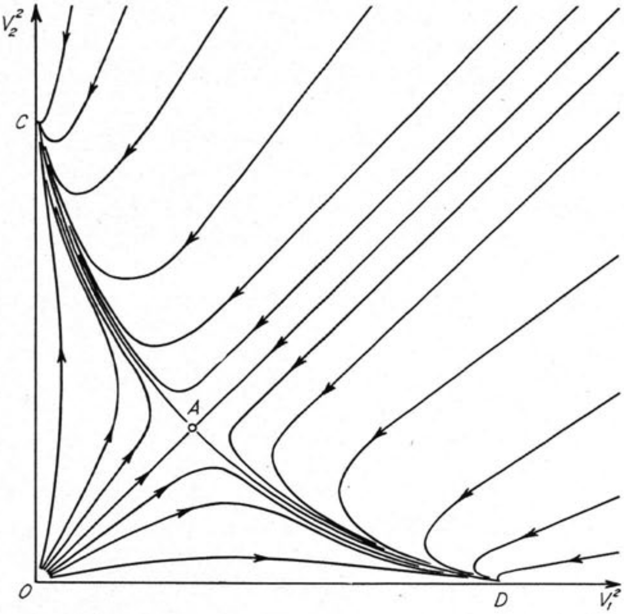

Fig. 6.3. Trajectories for Asynchronous Oscillator and $i = \alpha v + \beta v^2 + \gamma v^3$

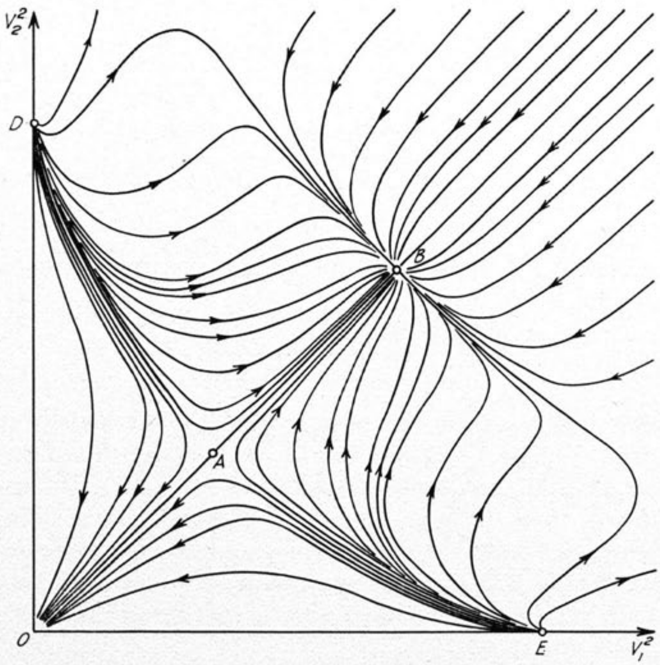

Fig. 6.4. Trajectories for Asynchronous Oscillator and $i = \alpha v + \beta v^2 + \gamma v^3 + \delta v^4 + \epsilon v^5$

The trajectories shown in Fig. 6.3 correspond to those obtained from such a polynomial. The point A shown in this figure is one that would ordinarily correspond to simultaneous oscillations, but it is a saddle point and is therefore unstable. On the other hand, if the current-voltage relationship is a fifth-degree polynomial and if certain other conditions are satisfied, then stable simultaneous oscillations are possible. Such a condition is represented by point B in Fig. 6.4. This figure also shows that as a rule asychronous simultaneous oscillations are not self-starting.

In general the representation of the synchronous oscillations in the V_1-V_2-ϕ space is not practicable. For certain conditions, however, Eq. 6.8 can be reduced to

$$\frac{d\phi}{dt} = f(V_1, V_2) \sin \phi \tag{6.11}$$

where $f(V_1, V_2) > 0$ for $V_1 > 0$, $V_2 > 0$. All trajectories will then approach the $\phi = \pi$ plane and a trajectory starting in this plane will remain in it. Figure 6.5 shows a typical group of trajectories in the $\phi = \pi$ plane. The simultaneous oscillations corresponding to point A in this figure are stable and self-starting.

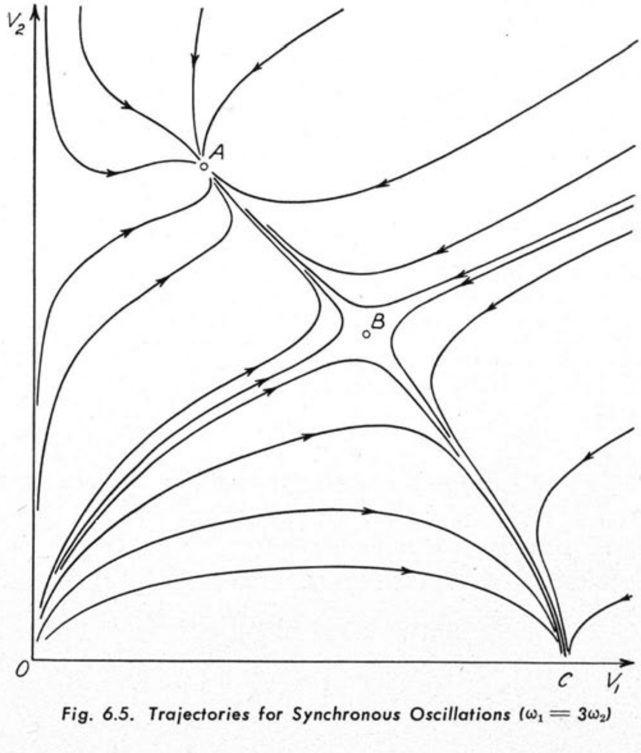

Fig. 6.5. Trajectories for Synchronous Oscillations ($\omega_1 = 3\omega_2$)

VII. THE LIMITATION OF THE AMPLITUDE OF OSCILLATIONS BY LAMPS

The amplitude of oscillations in an oscillator containing only linear elements will either decrease or increase indefinitely. It can reach a stationary value other than zero only if the oscillator contains a nonlinearity of some sort, in most cases an electron tube. The electron tube has a double function: it delivers a-c power to the passive elements and limits the amplitude of oscillation.

In the oscillator discussed here, these functions have been separated. The electron tube delivers the *a-c* power, but the amplitude is limited by a lamp inserted in the resonant circuit. The temperature of this lamp depends on the power dissipated in it and, hence, on the amplitude of oscillation. The resistance of the lamp varies as a function of filament temperature.

For small amplitudes of oscillation more power will be generated in the electron tube than is dissipated in the lamp, and as a consequence the amplitude will increase. Due to this increase, the temperature of the lamp, and also its resistance, will become greater. At a certain amplitude the resistance of the lamp will be so great that the power generated in the electron tube will be equal to that dissipated in the lamp, and steady-state conditions for amplitude are achieved.[14]

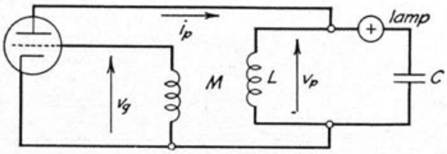

Fig. 7.1. Oscillator

An idealized oscillator using a lamp to limit the amplitude of oscillation is shown in Fig. 7.1. Actual oscillation using lamps to limit their amplitudes are usually more complex.[15, 16]

The variation in resistance of a conductor with the temperature is

$$R_{T_1} = R_{T_0}(1 + \alpha T_1) \tag{7.1}$$

where α is a constant depending on the material; R_{T_0} the resistance at the ambient temperature T_0; and $T_1 = T - T_0$ the difference between the temperature of the conductor and the ambient temperature.

It would be difficult to treat the circuit of Fig. 7.1 in full generality; therefore a number of simplifying assumptions must be made. It is assumed that no grid current flows and that only the linear portion of the triode characteristic is used. The plate-current is then a function of the plate voltage only:

$$\begin{aligned} i_p &= g_m\left(v_g + \frac{1}{\mu}v_p\right) \\ &= g_m\left(-\frac{M}{L} + \frac{1}{\mu}\right)v_p = -G_0 v_p \end{aligned} \tag{7.2}$$

where

$$G_0 = g_m\left(\frac{M}{L} - \frac{1}{\mu}\right)$$

is a constant with the dimensions of a conductance. Equation 7.2 shows that the triode of Fig. 7.1 can be replaced by the equivalent negative conductance $-G_0$. (Fig. 7.2) The lamp has been replaced in this figure by a resistance R_{T_1}.

The circuit of Fig. 7.2 can be further simplified. Since the frequency of oscillation ω differs by only a very small amount from $1/\sqrt{LC}$, the conductance $-G_0$ can be replaced by a resistance $-R_0$ in series with the inductance L (Fig. 7.3).

$$R_0 = \frac{L}{C}G_0$$

The total resistance of this resonant circuit is

$$\begin{aligned} R &= -R_0 + R_{T_1} \\ &= -R_0 + R_{T_0}(1 + \alpha T_1) \\ &= -R_1(1 - \beta T_1) \end{aligned} \tag{7.3}$$

where

$$\begin{aligned} R_1 &= R_0 - R_T \\ \beta &= \alpha\frac{R_{T_0}}{R_1} \end{aligned}$$

It is assumed that the quality factor Q of the circuit is high. The current i can then be approximated over a few cycles by

$$i = I\cos\omega t \tag{7.4}$$

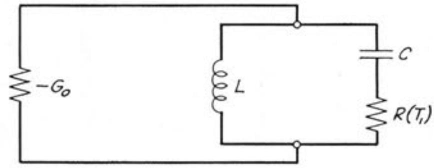

Fig. 7.2. Oscillator with Triode Replaced by Negative Conductance

The amplitude of current I will vary slowly and, for a short period of time, it can be assumed constant.

The total energy stored in the circuit of Fig. 7.3 is $LI^2/2$. During one cycle, it changes twice from electromagnetic to electrostatic energy and back again, but the total amount remains approximately constant. A slow change is caused, however, since some of the energy is dissipated (or generated) in the resistance R.

$$\frac{L}{2}\frac{dI^2}{dt} = \frac{R_1}{2} I^2 (1 - \beta T_1) \tag{7.5}$$

A similar differential equation can be found for the thermal energy mcT_1 of the lamp

$$mc\frac{dT_1}{dt} = -KT_1 + \frac{I^2}{2} R_{T_1} \tag{7.6}$$

where m is the mass of the heat element, c is the specific heat of the element, and K is the constant of thermal conductivity. For steady-state oscillations it is necessary that both I and T_1 remain constant; i.e.,

$$\frac{dI}{dt} = \frac{dT_1}{dt} = 0$$

The values of I and T_1 corresponding to steady-state oscillations are

$$I_0^2 = \frac{2K}{\beta R_0}; \qquad T_{1_0} = \frac{1}{\beta} \tag{7.7}$$

These oscillations may be stable or unstable depending on whether they will or will not resume their original amplitude after being subjected to a small disturbance. This can be determined by calculating and examining the conditions for stable steady-state oscillations from Eqs. 7.5 and 7.6. Such an examination shows that these oscillations are stable.

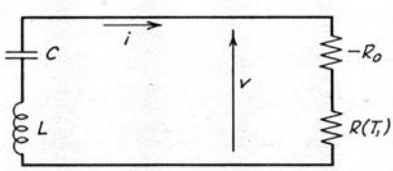

Fig. 7.3. Equivalent Circuit of Fig. 7.2

The behavior of the nonsteady-state oscillations can be determined also from Eqs. 7.5 and 7.6. Dividing one of these equations by the other eliminates time as a variable and makes possible the representation of the transient oscillations in a plane with I^2 and T_1 as the coordinates

$$\frac{dT_1}{dI^2} = \frac{L}{R_1 T} \frac{-T_1 + \dfrac{I^2}{2K}[R_0 - R_1(1 - \beta - T_1)]}{I^2(1 - \beta T_1)} \tag{7.8}$$

where $T = K/mc$ is the thermal time constant of the lamp.

This differential equation cannot be solved in closed form. An approximate solution, however, can be found by the method of isoclines. In Fig. 7.4, a typical solution is represented by a group of trajectories in a plane with the amplitude of oscillation as the abscissae and the temperature as the ordinates, the I^2-T_1 plane.

At a given time the oscillator will have an amplitude, I, and the lamp a temperature, T_1. The point in the I^2-T_1 plane defined by these two values will be called the "representative point" since it represents the state of the oscillator. As temperature and amplitude vary, the representative point will move along a line or "trajectory" in the I^2-T_1 plane. For example, if the oscillator starts from rest (point 0 in Fig. 7.4), it will move toward the point B along the dotted line.

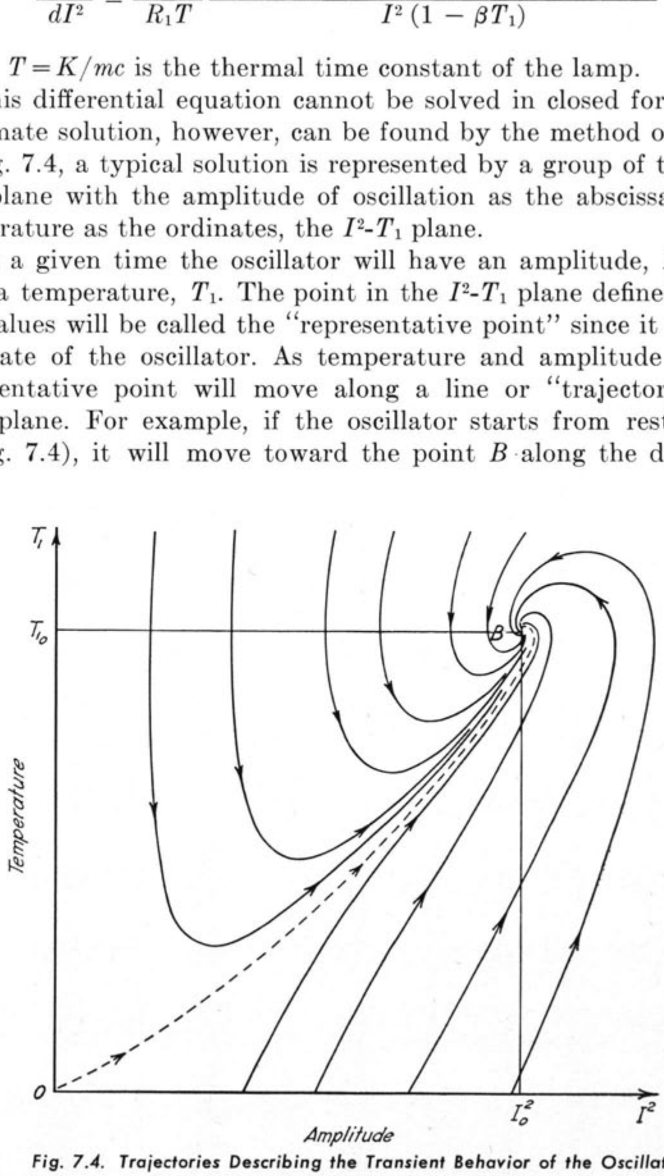

Fig. 7.4. Trajectories Describing the Transient Behavior of the Oscillator

The point B corresponds to steady-state oscillations with amplitude, I_0, and temperature, T_{1_0}.

The steady-state oscillations corresponding to point B will be disturbed constantly because of noise, variations in circuit parameters, and change of the ambient temperature; but it can be seen that the representative point, after having been displaced from B, will always return to its original position. Consequently the steady-state oscillations corresponding to point B are stable.

By a similar consideration it can be seen that point A is unstable since the representative point will never return to this point after it has been given a small displacement.

By assuming the current i to be

$$i = I \cos \omega t$$

it was possible to describe the oscillator by the equations 7.5 and 7.6. If no assumption regarding the current i is made, however, it is necessary to use three differential equations of the first order. For example, the circuit of Fig. 7.3 leads to

$$\begin{aligned} \frac{di}{dt} &= y \\ \frac{dy}{dt} &= \frac{1}{L} R_1 (1 - \beta T_1)\, y - \frac{1}{LC} i \\ \frac{dT_1}{dt} &= \frac{T_1}{T} + \frac{1}{mc} (R_0 - R_1 (1 - \beta T_1))\, i^2 \end{aligned} \tag{7.9}$$

where i is the current through the resonant circuit.

Similar to the I^2-T_1 plane it is possible to set up a i-y-T_1 space. The solutions of Eqs. 7.9 could then be represented by a group of trajectories in this space.

VIII. CONCLUSIONS

Several oscillatory systems have been discussed in this bulletin with special emphasis on nonsteady-state or "transient" oscillations. For autonomous systems with one degree of freedom and any degree of nonlinearity, a complete representation of the transient oscillations is possible in the phase plane. For more complicated systems, this method of representation is either not possible or impracticable since it leads to spaces with four or more dimensions.

A representation in a plane (which is no longer the phase plane) is, however, still possible in some important cases if it can be assumed that the oscillations are almost sinusoidal. As has been shown, the transient oscillations can then often be described by two first order differential equations:

$$\frac{dX}{dt} = f_1(X,Y)$$

$$\frac{dY}{dt} = f_2(X,Y) \tag{8.1}$$

where X and Y may be an amplitude, phase angle, temperature, etc. These equations do not give a complete description of the oscillations, however, since they neglect the higher harmonics. This does not lead to appreciable errors if the losses in the resonant circuits are small (high Q) since then the amplitudes of the higher harmonics are also small. Even for low Q circuits the results calculated from Eq. 8.1 are confirmed surprisingly well by experiments.

The general procedure is to eliminate from Eqs. 8.1 by division:

$$\frac{dX}{dY} = \frac{f_1(X,Y)}{f_2(X,Y)} \tag{8.2}$$

The solutions of this differential equation can then be represented as a group of trajectories in the X-Y plane, in which an oscillation with parameters X and Y corresponds to a point lying on one of the trajectories in this plane. If the system is not disturbed, then this point will move along the trajectories defined by Eq. 8.2. Steady-state oscillations correspond to singular points of this equation. These oscillations are stable if after being subjected to a small disturbance the parameters X and Y will return to this point. Besides the singular points, the X-Y plane may also contain closed trajectories or drift curves.

Four different oscillatory phenomena were treated by this method: parametric excitation, synchronization, simultaneous oscillations and amplitude limitation by means of lamps. The method can be applied to a number of other problems, such as oscillations in conservative systems with several degrees of freedom, self-modulating oscillators, and the motor-boating effect occurring in audio oscillators.

The results obtained in Chapter V and VI were confirmed experimentally. The degree of correspondence between theory and experiment was good.

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